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On the paradox of heat conduction in porous media subject to lack of local thermal equilibrium

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Abstract

An apparent paradox that appears in problems of heat conduction in porous media subject to lack of local thermal equilibrium (LaLotheq) that was introduced by Vadasz [Transport in Porous Media 59 (2005) 341–355] is reformulated and resolved. This apparent paradox relates to a combination of Dirichlet and insulation boundary conditions and leads the solution towards local thermal equilibrium (Lotheq).
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Keywords: Porous media; Heat conduction; Local thermal equilibrium; Local thermal non-equilibrium; Lotheq; LaLotheq

1. Introduction

An apparent paradox related to the problem of heat conduction in porous media subject to lack of local thermal equilibrium (LaLotheq) was introduced by Vadasz [1]. The said paradox arises for a combination of Dirichlet and insulation boundary conditions.

Previous work on porous media heat transfer subject to lack of local thermal equilibrium (LaLotheq) was undertaken among others by Nield [2,3], Minkowycz et al. [4], Banu and Rees [5], Baytas and Pop [6], Kim and Jang [7], Rees [8], Alazmi and Vafai [9], and Nield et al. [10]. In particular, Nield [3] shows that for uniform thermal conductivities the steady state conduction leads to local thermal equilibrium (Lotheq) if the temperature or its normal derivative on the boundary are identical for both phases.

Tzou [11,12] refers to experimental results in porous media heat conduction identifying thermal oscillations and overshooting, and explains them by applying the dual-phase-lagging (DuPhlag) model. In particular, Minkowycz et al. [4] link the LaLotheq model with the DuPhlag model in a similar manner to Tzou [11,12] however they do not claim the possibility of oscillations. Vadasz [13–15] proved that such oscillations are not possible.

The present paper aims at demonstrating that, for a fluid saturated porous layer subject to heat conduction (transient as well as steady state) and any combination of imposed temperatures and insulation on the boundary, the dual-phase thermal conduction leads apparently back to Lotheq conditions and to a very particular case of identical effective thermal diffusivities for both phases. This paradox is resolved in this paper. While Vadasz [1] introduced the apparent paradox in terms of a three-dimensional general domain the present paper deals with a two-dimensional rectangular domain and the generalization to any three-dimensional domain is discussed. While the paper is particularly aimed at the conditions applicable to a porous medium Vadasz [15,16] showed that similar results and conclusions are applicable to suspensions of solid particles in fluids, or to bi-composite media (a combination of two different solid phases).

In the present paper a *contextual notation* is introduced to distinguish between dimensional and dimensionless

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Nomenclature

Fh_s	dimensionless group defined in Eq. (14)
Fh_f	dimensionless group defined in Eq. (14)
Fo_q	heat flux related Fourier number, equals $\alpha_e\tau_q/L^2$
Fo_T	temperature gradient related, Fourier number, equals $\alpha_e\tau_T/L^2$
Bh	bi-harmonic dimensionless group, equals β_e/L^2
Bf	dimensionless group, equals $Bh/Fo_q = \alpha_s\alpha_f/\alpha_e^2$
h	integral heat transfer coefficient for the heat conduction at the solid–fluid interface (dimensional)
k_s	effective thermal conductivity of the solid phase, equals $(1-\varphi)\tilde{k}_s$ (dimensional)
$\tilde{k}_s$	thermal conductivity of the solid phase (dimensional)
k_f	effective thermal conductivity of the fluid phase, equals $\varphi\tilde{k}_f$ (dimensional)
$\tilde{k}_f$	thermal conductivity of the fluid phase (dimensional)
L	the length of the porous slab (dimensional)
Ni_s	solid phase Nield number, Eq. (14)
Ni_f	fluid phase Nield number, Eq. (14)
$\boldsymbol{q}_*$	heat flux vector (dimensional)
t_*	time (dimensional)
T	temperature (dimensional)
T_C	coldest wall temperature (dimensional)
T_H	hottest wall temperature (dimensional)
x_*	horizontal co-ordinate (dimensional)
$\boldsymbol{x}_*$	spatial variables vector (dimensional) equals (x_*, y_*, z_*)

Greek symbols

α_e	effective thermal diffusivity, defined in Eq. (6) (dimensional)
α_s	solid phase effective thermal diffusivity, equals k_s/γ_s (dimensional)
α_f	fluid phase effective thermal diffusivity, equals k_f/γ_f (dimensional)
β_e	bi-harmonic coefficient, defined in Eq. (6) (dimensional)
γ_s	solid phase effective heat capacity, equals $(1-\varphi)\rho_s c_s$ (dimensional)
γ_f	fluid phase effective heat capacity, equals $\varphi\rho_f c_{p,f}$ (dimensional)
θ_i	dimensionless temperature, equals $(T_i - T_C)/(T_H - T_C)$ for $i = s, f$
η_γ	heat capacities ratio, equals γ_f/γ_s
η_k	thermal conductivity ratio, equals k_f/k_s
φ	porosity
ψ	dimensionless group, equals $Fo_T/Fo_q = \tau_T/\tau_q$
ρ^s	solid phase density
ρ^f	fluid phase density
τ_q	time lag associated with the heat flux, defined in Eq. (6) (dimensional)
τ_T	time lag associated with the temperature gradient, defined in Eq. (6) (dimensional)

Subscripts

$*$	corresponding to dimensional values
s	related to the solid phase
f	related to the fluid phase

variables and parameters. The *contextual notation* implies that an asterisk subscript is used to identify dimensional variables and parameters only when ambiguity may arise when the asterisk subscript is not used. For example x_* is the dimensional horizontal coordinate, while x is its corresponding dimensionless counterpart. However k_s is the effective solid phase thermal conductivity, a dimensional parameter that appears without an asterisk subscript without causing ambiguity.

2. Problem formulation and the apparent paradox

Let us consider the heat conduction in a rectangular two-dimensional fluid saturated porous domain that is exposed to different constant temperatures on the vertical walls and to insulation conditions on the horizontal walls as presented in Fig. 1. Heat conduction in porous media subject to lack of local thermal equilibrium (LaLotheq) is governed at the macro-level by the following equations that represent averages over each phase within an REV (representative elementary volume)

$$\gamma_s \frac{\partial T_s}{\partial t_*} = k_s \nabla_*^2 T_s - h(T_s - T_f) \tag{1}$$

$$\gamma_f \frac{\partial T_f}{\partial t_*} = k_f \nabla_*^2 T_f + h(T_s - T_f) \tag{2}$$

where $Q_{sf} = h(T_s - T_f)$ represents the rate of heat generation in the fluid phase within the REV due to the heat transferred over the fluid–solid interface, and where $\gamma_s = (1-\varphi)\rho_s c_s$ and $\gamma_f = \varphi\rho_f c_{p,f}$ are the solid phase and fluid phase effective heat capacities, respectively, φ is the porosity, $k_s = (1-\varphi)\tilde{k}_s$ and $k_f = \varphi\tilde{k}_f$ are the effective thermal conductivities of the solid and fluid phases, respectively. The coefficient $h > 0$, carrying units of $W\,m^{-3}\,K^{-1}$, is a macro-level integral heat transfer coefficient for the heat conduction at the fluid–solid interface (averaged over the REV) that is assumed independent of the phases' temperatures and independent of time. Note that this coefficient is conceptually distinct from the convection heat transfer coefficient and is anticipated to depend on the thermal conductivities of both phases as well as on the surface area to volume ratio (specific area) of the medium [16].

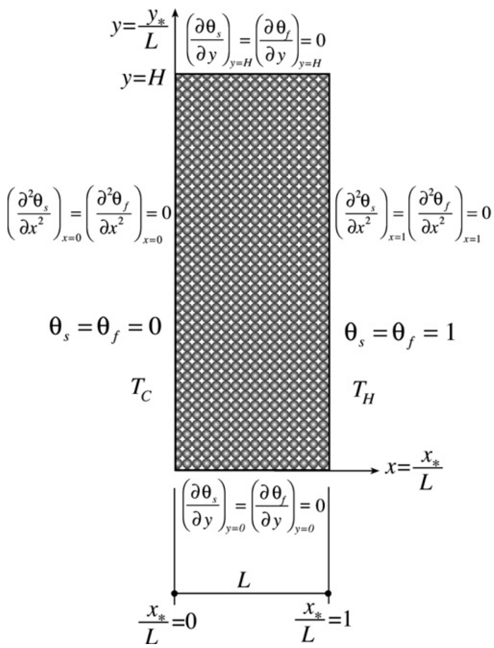

Fig. 1. Problem formulation – heat conduction in a two-dimensional rectangular domain subject to lack of local thermal equilibrium (LaLotheq).

Eqs. (1) and (2) are linearly coupled and represent the traditional form of expressing the process of heat conduction in porous media subject to LaLotheq [3,17]. When the value of the interface heat transfer coefficient vanishes, $h = 0$ (physically representing an interface that is an ideal insulator, e.g. the solid–fluid interface is coated with a highly insulating material) Eqs. (1) and (2) un-couple and the solution for the temperature of each phase is independent of the other phase, the phase having the highest thermal diffusivity producing a temperature that equilibrates faster to its steady state value. Very large values of h on the other hand lead to local thermal equilibrium (Lotheq) as observed by dividing Eqs. (1) and (2) by h and looking for the limit as $h \to \infty$ that produces (at least at the leading order) $T_s = T_f$. The latter occurs because despite the fact that one phase (the slow one) diffuses heat at a slower pace a perfect compensation occurs due to the interface heat transfer, i.e. the change in temperature in the faster phase is instantly converted into an identical temperature change in the slower phase via the heat transferred through the interface without any resistance because $h \to \infty$. Similar results may be obtained with a finite interface heat transfer coefficient, h, if the thermal diffusivities of both phases are identical, i.e. $\alpha_s = (k_s/\gamma_s) = (k_f/\gamma_f) = \alpha_f$. Then, both phases will diffuse heat at the same pace leading naturally to $T_s = T_f$ and a vanishing heat transfer over the interface $h(T_s - T_f) = 0$ irrespective of the value of h.

For the two-dimensional system considered here (see Fig. 1) the Laplacian operator ∇_*^2 is defined in the form $\nabla_*^2 = \partial^2/\partial x_*^2 + \partial^2/\partial y_*^2$. The boundary conditions applicable to the problem at hand are constant temperature at the vertical walls and insulation at the top and bottom horizontal walls

$$x_* = 0 : T_s = T_f = T_C \qquad \text{(a)}$$
$$x_* = L : T_s = T_f = T_H \qquad \text{(b)}$$
$$y_* = 0 \text{ and } y_* = H_* : \left(\frac{\partial T_s}{\partial y_*}\right)_{y_*=0,H_*} = \left(\frac{\partial T_f}{\partial y_*}\right)_{y_*=0,H_*} = 0 \qquad \text{(c)}. \tag{3}$$

The initial conditions are related to the initial physical conditions of having the porous medium in thermal equilibrium with its surroundings leading to the same uniform constant temperature for both phases, i.e.

$$t_* = 0 : (T_s)_{t_*=0} = (T_f)_{t_*=0} = T_0 = \text{constant}. \tag{4}$$

Two methods are in principle available to solving the problem (1) and (2) analytically subject to the boundary conditions (3) and initial conditions (4). The first method ("the eigenvectors method") is linked to evaluating the eigenvalues and eigenvectors directly from system (1) and (2). The second method ("the elimination method") is related to deriving an equivalent equation that is second order in time and fourth order in space via elimination of the dependent variables T_s and T_f. The first step in this paper is to present the paradox that was introduced by Vadasz [1], then the solution is presented via both methods listed above and their results compared.

The stated paradox appears when attempting to solve the problem via "the elimination method". The elimination of the dependent variables T_s and T_f (one at a time or simultaneously) is accomplished via one of the two methods presented by Vadasz [1,13,16] leading to two independent equations for each phase in the form

$$\tau_q \frac{\partial^2 T_i}{\partial t_*^2} + \frac{\partial T_i}{\partial t_*} = \alpha_e \left[\nabla_*^2 T_i + \tau_T \nabla_*^2 \left(\frac{\partial T_i}{\partial t_*}\right) - \beta_e \nabla_*^4 T_i\right] \quad \forall i = s, f \tag{5}$$

where the index i can take the values s representing the solid phase or f standing for the fluid phase and where the following notation was used

$$\tau_q = \frac{\gamma_s \gamma_f}{h(\gamma_s + \gamma_f)}; \quad \alpha_e = \frac{(k_s + k_f)}{(\gamma_s + \gamma_f)};$$
$$\tau_T = \frac{(\gamma_s k_f + \gamma_f k_s)}{h(k_s + k_f)}; \quad \beta_e = \frac{k_s k_f}{h(k_s + k_f)} \tag{6}$$

(Note: the present definition of β_e is different than in Vadasz [1]). Eq. (5) is a linear equation that applies to each phase, while its parameters are effective coefficients common to both phases. By imposing the combination of Dirichlet (constant temperatures) and insulation boundary conditions expressed by Eq. (3) and assuming uniform and identical initial conditions for both phases expressed by Eq. (4) provides two boundary conditions in each direction and one initial condition for each phase. However, Eq. (5) is fourth order in space and second order in time, requiring therefore two additional boundary conditions in each direction and one additional initial condition. The latter conditions can be derived from the original ones (3) and

(4) by using the original equations (1) and (2), which govern the heat conduction at all times (including $t_* = 0$) and over the whole physical domain including the boundaries. The derived boundary and initial conditions to be used in connection with the solution to Eq. (5) are

$$\begin{aligned} &x_* = 0: T_i = T_C;\ \ (\partial^2 T_i/\partial x_*^2)_{x_*=0} = 0\ \ \forall i = \mathrm{s,f} \quad \text{(a)}\\ &x_* = L: T_i = T_H;\ \ (\partial^2 T_i/\partial x_*^2)_{x_*=L} = 0\ \ \forall i = \mathrm{s,f} \quad \text{(b)}\\ &y_* = 0, H_*: (\partial T_i/\partial y_*)_{y_*=0,H_*} = 0;\\ &(\partial^3 T_i/\partial y_*^3)_{y_*=0,H_*} = 0\ \ \forall i = \mathrm{s,f} \quad \text{(c)} \end{aligned} \tag{7}$$

$$\begin{aligned} &t_* = 0: (T_i)_{t_*=0} = T_0 = \text{constant};\\ &(\partial T_i/\partial t_*)_{t_*=0} = 0 \quad \forall i = \mathrm{s,f} \end{aligned} \tag{8}$$

Eq. (5) that is identical for both phases shares common effective parameters for both phases, solved subject to identical boundary and initial conditions for each phase, (7) and (8) produce therefore a solution that is expected to be identical for both phases, i.e.

$$T_s(t_*, \mathbf{x}_*) = T_f(t_*, \mathbf{x}_*) \quad \forall (t_* \geqslant 0, x_* \in [0, L], y_* \in [0, H_*]) \tag{9}$$

where $\mathbf{x}_* = (x_*, y_*)$ represents the spatial variables. Eq. (9) is identified as the requirement for local thermal equilibrium (Lotheq) in porous media conduction causing the heat generation due to the heat transfer at the fluid–solid interface $Q_{sf} = h(T_s - T_f)$ to vanish. It was obtained accurately from the original system of Eqs. (1) and (2) subject to the specified boundary and initial conditions and other than that no other imposed restrictions. This result is quite astonishing and intriguing because it suggests that *local thermal equilibrium (Lotheq) exists naturally in any porous domain subject to heat conduction and a combination of constant temperature and insulation boundary conditions*. However, this conclusion needs further investigation. Substituting Eq. (9) into Eqs. (1) and (2) yields

$$\frac{\partial T_s}{\partial t_*} = \alpha_s \nabla_*^2 T_s \tag{10}$$

$$\frac{\partial T_f}{\partial t_*} = \alpha_f \nabla_*^2 T_f \tag{11}$$

where $\alpha_s = k_s/\gamma_s$ and $\alpha_f = k_f/\gamma_f$. The solution to Eqs. (10) and (11) subject to the same boundary and initial conditions as indicated in Eqs. (3) and (4) has to be identical to the corresponding solutions of Eq. (5) subject to the equivalent boundary and initial conditions (7) and (8), respectively. This means that Eqs. (10) and (11) are expected to produce an identical solution $T_s(t_*, \mathbf{x}_*) = T_f(t_*, \mathbf{x}_*)\ \forall (t_* \geqslant 0, x_* \in [0, L], y_* \in [0, H_*])$ despite the fact that in general their respective thermal diffusivities may vary substantially. The latter cannot be accomplished unless $\alpha_s = \alpha_f$, leading to the inevitable conclusion that consistency requires the effective thermal diffusivities of both phases to be identical. The latter condition was not explicitly imposed *a priori*, nor implied in any of the subsequent derivations. Nevertheless, it was obtained as a result that is linked to the consequences of Eq. (9). However, the effective thermal diffusivities of both phases are based on material properties and therefore this limitation cannot generally be applicable. We must therefore insist that $\alpha_s \neq \alpha_f$ in which case Eqs. (10) and (11) subject to the boundary and initial conditions (3) and (4) will produce distinct solutions $T_s(t_*, \mathbf{x}_*) \neq T_f(t_*, \mathbf{x}_*)$ leading back to Eqs. (1) and (2) with non-vanishing interphase heat transfer $Q_{sf} = h(T_s - T_f) \neq 0$ and the whole process cycles indefinitely introducing the paradox.

3. Solution by the eigenvectors method

The system of Eqs. (1) and (2) and its corresponding boundary and initial conditions are rendered dimensionless by using L to scale the space variables x_* and y_*, in the form $x = x_*/L, y = y_*/L$, L^2/α_e to scale time, that is, $t = t_*\alpha_e/L^2$ and introducing the dimensionless temperature $\theta_i = (T_i - T_C)/(T_H - T_C)\ \forall i = \mathrm{s,f}$, leading to the following dimensionless form of Eqs. (1) and (2)

$$Fh_s \frac{\partial \theta_s}{\partial t} = \frac{1}{Ni_s} \nabla^2 \theta_s - (\theta_s - \theta_f) \tag{12}$$

$$Fh_f \frac{\partial \theta_f}{\partial t} = \frac{1}{Ni_f} \nabla^2 \theta_f + (\theta_s - \theta_f) \tag{13}$$

where the following dimensionless groups listing the solid phase and fluid phase Nield numbers $Ni_s Ni_f$, respectively, and additional dimensionless groups that emerged

$$\begin{aligned} &Bh = \frac{\beta_e}{L^2};\ \ Ni_f = \frac{hL^2}{k_f};\ \ Ni_s = \frac{hL^2}{k_s};\ \ Fh_f = \frac{(\gamma_s + \gamma_f)}{\gamma_s} Fo_q = \frac{\alpha_e \gamma_f}{hL^2};\\ &Fh_s = \frac{(\gamma_s + \gamma_f)}{\gamma_f} Fo_q = \frac{\alpha_e \gamma_s}{hL^2};\ \ Fo_q = \frac{\alpha_e \tau_q}{L^2};\ \ Fo_T = \frac{\alpha_e \tau_T}{L^2} \end{aligned} \tag{14}$$

The dimensionless form of the boundary and initial conditions (3) and (4) are

$$\begin{aligned} &x = 0: \theta_i = 0 \quad \forall i = \mathrm{s,f} \quad \text{(a)}\\ &x = 1: \theta_i = 1 \quad \forall i = \mathrm{s,f} \quad \text{(b)}\\ &y = 0, H: (\partial \theta_i/\partial y)_{y=0,H} = 0 \quad \forall i = \mathrm{s,f} \quad \text{(c)} \end{aligned} \tag{15}$$

$$t = 0: (\theta_i)_{t=0} = \theta_0 = \text{constant} \quad \forall i = \mathrm{s,f} \tag{16}$$

The solution to Eqs. (12) and (13) is separated into steady state $\theta_{i,sts}$ and transient $\theta_{i,tr}$ parts in the form $\theta_i = \theta_{i,sts} + \theta_{i,tr}$. The steady state for both phases $i = \mathrm{s,f}$ is satisfied by the linear solution $\theta_{i,sts} = x$, which satisfies the boundary conditions (15). It is sensible to assume for $\theta_{i,tr}$ to be independent of the y coordinate and this assumption satisfies the boundary conditions (15c) at $y = 0, H$. As a result, the equations governing the transient have the form

$$Fh_s \frac{\partial \theta_{s,tr}}{\partial t} = \frac{1}{Ni_s} \frac{\partial^2 \theta_{s,tr}}{\partial x^2} - (\theta_{s,tr} - \theta_{f,tr}) \tag{17}$$

$$Fh_f \frac{\partial \theta_{f,tr}}{\partial t} = \frac{1}{Ni_f} \frac{\partial^2 \theta_{f,tr}}{\partial x^2} + (\theta_{s,tr} - \theta_{f,tr}) \tag{18}$$

subject to boundary and initial conditions that are obtained following the substitution of $\theta_i = x + \theta_{i,\mathrm{tr}}$ into (15) and (16) leading to

$$x = 0, 1: \theta_{i,\mathrm{tr}} = 0 \quad \forall i = \mathrm{s}, \mathrm{f} \tag{19}$$

$$t = 0: (\theta_{i,\mathrm{tr}})_{t=0} = \theta_0 - x \quad \forall i = \mathrm{s}, \mathrm{f} \tag{20}$$

The solutions to Eqs. (17) and (18) subject to the boundary and initial conditions (19) and (20) are obtained via separation of variables in the form of two equations for each phase in the form $\theta_{i,\mathrm{tr}} = \phi_{in}(t)u_n(x)$ where the functions $u_n(x)$ are identical for both phases because they satisfy the same equations and the same boundary conditions. The latter statement about the fact that both phases share the same eigenfunctions $u_n(x)$ can be proven in detail, a step that is skipped here for brevity of the presentation. The resulting equation is $\mathrm{d}^2u_n/\mathrm{d}x^2 + \kappa_n^2 u_n = 0$. The solution to this equation subject to the homogeneous boundary conditions derived from (19) $(u_n)_{x=0,1} = 0$ and $(\mathrm{d}^2u_n/\mathrm{d}x^2)_{x=0,1} = 0$ at $x = 0, 1$, is $u_n = \sin(\kappa_n x)$, and the resulting eigenvalues are $\kappa_n = n\pi \ \forall n = 1, 2, 3, \ldots$. Substituting this eigenfunction solution into Eqs. (17) and (18) yields the following set of ordinary differential equations for the eigenfunctions in the time domain $\phi_{\mathrm{s}n}(t)$ and $\phi_{\mathrm{f}n}(t)$

$$\begin{cases} \dfrac{\mathrm{d}\phi_{\mathrm{s}n}}{\mathrm{d}t} = a_n\phi_{\mathrm{s}n} + b\phi_{\mathrm{f}n} & \text{(a)} \\ \dfrac{\mathrm{d}\phi_{\mathrm{f}n}}{\mathrm{d}t} = c\phi_{\mathrm{s}n} + d_n\phi_{\mathrm{f}n} & \text{(b)} \end{cases} \tag{21}$$

where the definition of the coefficients that emerged from the substitution is

$$a_n = -\frac{(n^2\pi^2 + Ni_{\mathrm{s}})}{Ni_{\mathrm{s}}Fh_{\mathrm{s}}}; \quad d_n = -\frac{(n^2\pi^2 + Ni_{\mathrm{f}})}{Ni_{\mathrm{f}}Fh_{\mathrm{f}}};$$
$$b = Fh_{\mathrm{s}}^{-1}; \quad c = Fh_{\mathrm{f}}^{-1} \tag{22}$$

The general solution has therefore the form

$$\theta_i = x + \sum_{n=1}^{\infty} \phi_{in}(t)\sin(n\pi x) \quad \forall i = \mathrm{s}, \mathrm{f} \tag{23}$$

where $\phi_{in}(t)$ are the solutions to the system of Eq. (21). However, the system (21) needs initial conditions in terms of $\phi_{\mathrm{s}n}(0)$ and $\phi_{\mathrm{f}n}(0)$. The latter may be obtained from the initial conditions of θ_{s} and θ_{f}, (16), applied to Eq. (23) in the form

$$(\theta_i)_{t=0} \equiv x + \sum_{n=1}^{\infty} \phi_{in}(0)\sin(n\pi x) = \theta_0 \quad \forall i = \mathrm{s}, \mathrm{f} \tag{24}$$

Multiplying (24) by $\sin(j\pi x)$, integrating the result over the whole domain, i.e. $\int_0^1(\cdot)\,\mathrm{d}x$, and using the orthogonality conditions yields an identical initial condition for both phases, $\phi_{in}(0)$, in the form

$$\phi_{\mathrm{s}n}(0) = \phi_{\mathrm{f}n}(0) = \phi_{no} = \frac{2\{(-1)^n + [1 - (-1)^n]\theta_0\}}{n\pi}. \tag{25}$$

The eigenvalues and eigenvectors are obtained from Eq. (21) to yield

$$\lambda_{1,2n} = \frac{1}{2}\left[(a_n + d_n) \pm \sqrt{(a_n + d_n)^2 - 4(a_nd_n - bc)}\right] \tag{26}$$

$$\mathbf{V}_{1n} = [1, (\lambda_{1n} - a_n)/b]^{\mathrm{T}} = [1, c/(\lambda_{1n} - d_n)]^{\mathrm{T}} \tag{27}$$

$$\mathbf{V}_{2n} = [1, (\lambda_{2n} - a_n)/b]^{\mathrm{T}} = [1, c/(\lambda_{2n} - d_n)]^{\mathrm{T}} \tag{28}$$

and the solution in terms of these eigenvectors has the form

$$\boldsymbol{\phi}_{\boldsymbol{n}} = C_1\mathbf{V}_{\boldsymbol{1n}}\mathrm{e}^{\lambda_{1n}t} + C_2\mathbf{V}_{\boldsymbol{2n}}\mathrm{e}^{\lambda_{2n}t} \tag{29}$$

where $\boldsymbol{\phi}_{\boldsymbol{n}} = [\phi_{\mathrm{s}n}, \phi_{\mathrm{f}n}]^{\mathrm{T}}$.

The following relationships obtained from (26)–(28) are useful in the following analysis. From (26) one may obtain (see Appendix for details)

$$(\lambda_{1n} - d_n) = -(\lambda_{2n} - a_n) \text{ or } (\lambda_{1n} - a_n) = -(\lambda_{2n} - d_n) \tag{30}$$

From Eqs. (27) and (28) one gets

$$(\lambda_{1n} - a_n)(\lambda_{1n} - d_n) = bc, \text{ and } (\lambda_{2n} - a_n)(\lambda_{2n} - d_n) = bc \tag{31}$$

respectively. The following identities that are obtained from (14) and (6) are useful to demonstrate the next point

$$(Ni_{\mathrm{s}}Ni_{\mathrm{f}}Fh_{\mathrm{s}}Fh_{\mathrm{f}})^{-1} = Bh/Fo_q = Bf;$$
$$(Ni_{\mathrm{s}} + Ni_{\mathrm{f}})(Ni_{\mathrm{s}}Ni_{\mathrm{f}}Fh_{\mathrm{s}}Fh_{\mathrm{f}})^{-1} = Fo_q^{-1} \tag{32}$$

By substituting (22) and the definitions (14) and (6) yields (introducing the notation of v_n and ω_n^2)

$$v_n = -(a_n + d_n) = \left[n^2\pi^2\frac{(\alpha_{\mathrm{s}} + \alpha_{\mathrm{f}})}{\alpha_{\mathrm{e}}} + \frac{1}{Fo_q}\right] \tag{33}$$

and using also (32) leads to

$$\omega_n^2 = (a_nd_n - bc) = \frac{n^4\pi^4}{Ni_{\mathrm{s}}Ni_{\mathrm{f}}Fh_{\mathrm{s}}Fh_{\mathrm{f}}} + \frac{(Ni_{\mathrm{s}} + Ni_{\mathrm{f}})}{Ni_{\mathrm{s}}Ni_{\mathrm{f}}Fh_{\mathrm{s}}Fh_{\mathrm{f}}}n^2\pi^2$$
$$= \left(\frac{Bh}{Fo_q}\right)n^4\pi^4 + \frac{n^2\pi^2}{Fo_q}. \tag{34}$$

By using the initial conditions (25) into Eq. (29) and evaluating the coefficients C_1 and C_2 produces the solutions in the time domain $\phi_{in}(t)$ needed in the general solution (23), in the form

$$\phi_{\mathrm{s}n} = \frac{[\lambda_{2n} - (a_n + b)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{1n}t} - \frac{[\lambda_{1n} - (a_n + b)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{2n}t}$$
$$\phi_{\mathrm{f}n} = \frac{(\lambda_{1n} - a_n)[\lambda_{2n} - (a_n + b)]\phi_{no}}{b(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{1n}t}$$
$$- \frac{(\lambda_{2n} - a_n)[\lambda_{1n} - (a_n + b)]\phi_{no}}{b(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{2n}t} \tag{35}$$

By using now Eq. (30) followed by using (31) yields

$$\phi_{\mathrm{f}n} = \frac{[\lambda_{2n} - (d_n + c)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{1n}t} - \frac{[\lambda_{1n} - (d_n + c)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{2n}t} \tag{36}$$

where ϕ_{no} is defined in Eq. (25), λ_{1n} and λ_{2n} are defined in Eq. (26), and a_n, b, c, d_n are defined in Eq. (22). Substituting (35) and (36) into (23) produces the general solution obtained via the eigenvectors method in the form

$$\theta_s = x + \sum_{n=1}^{\infty}\left\{\frac{[\lambda_{2n}-(a_n+b)]\phi_{no}}{(\lambda_{2n}-\lambda_{1n})}e^{\lambda_{1n}t} - \frac{[\lambda_{1n}-(a_n+b)]\phi_{no}}{(\lambda_{2n}-\lambda_{1n})}e^{\lambda_{2n}t}\right\}\sin(n\pi x) \quad (37)$$

$$\theta_f = x + \sum_{n=1}^{\infty}\left\{\frac{[\lambda_{2n}-(d_n+c)]\phi_{no}}{(\lambda_{2n}-\lambda_{1n})}e^{\lambda_{1n}t} - \frac{[\lambda_{1n}-(d_n+c)]\phi_{no}}{(\lambda_{2n}-\lambda_{1n})}e^{\lambda_{2n}t}\right\}\sin(n\pi x) \quad (38)$$

4. Solution by the elimination method

Eq. (5) and its corresponding boundary and initial conditions are converted into a dimensionless form by using the same scales introduced in the previous section, leading to

$$Fo_q\frac{\partial^2\theta_i}{\partial t^2} + \frac{\partial\theta_i}{\partial t} = \nabla^2\theta_i + Fo_T\nabla^2\left(\frac{\partial\theta_i}{\partial t}\right) - Bh\nabla^4\theta_i \quad \forall i = s, f \quad (39)$$

where the two Fourier numbers, Fo_q, Fo_T, and one additional dimensionless group (the bi-harmonic number, Bh) that were defined in Eq. (14) emerged. The dimensionless form of the boundary and initial conditions (7) and (8) that are required for the solution of Eq. (39) are

$$\begin{aligned} &x = 0: \theta_i = 0; \quad (\partial^2\theta_i/\partial x^2)_{x=0} = 0 \quad \forall i = s, f \qquad (a)\\ &x = 1: \theta_i = 1; \quad (\partial^2\theta_i/\partial x^2)_{x=1} = 0 \quad \forall i = s, f \qquad (b)\\ &y = 0, H: (\partial\theta_i/\partial y)_{y=0,H} = 0;\\ &\quad (\partial^3\theta_i/\partial y^3)_{y=0,H} = 0 \quad \forall i = s, f \qquad (c) \end{aligned} \quad (40)$$

$$t = 0: (\theta_i)_{t=0} = \theta_0 = \text{constant}; \quad (\partial\theta_i/\partial t)_{t=0} = 0 \quad \forall i = s, f \quad (41)$$

The solution to Eq. (39) is separated into steady state $\theta_{i,sts}$ and transient $\theta_{i,tr}$ parts in the form $\theta_i = \theta_{i,sts} + \theta_{i,tr}$. The steady state for both phases $i = s, f$ is satisfied by the linear solution $\theta_{i,sts} = x$, which satisfies the boundary conditions (40). In addition, it is a sensible assumption for the transient part $\theta_{i,tr}$ to be considered independent of the y coordinate and this assumption satisfies the boundary conditions (40c) at $y = 0, H$. As a result, the equation governing the transient has the form

$$Fo_q\frac{\partial^2\theta_{i,tr}}{\partial t^2} + \frac{\partial\theta_{i,tr}}{\partial t} = \frac{\partial^2\theta_{i,tr}}{\partial x^2} + Fo_T\frac{\partial^3\theta_{i,tr}}{\partial t\,\partial x^2} - Bh\frac{\partial^4\theta_{i,tr}}{\partial x^4} \quad \forall i = s, f \quad (42)$$

subject to boundary and initial conditions that are obtained following the substitution of $\theta_i = x + \theta_{i,tr}$ $\forall i = s, f$ into (40) and (41) leading to

$$\begin{aligned} &x = 0: \theta_{i,tr} = 0; \quad (\partial^2\theta_{i,tr}/\partial x^2)_{x=0} = 0 \quad \forall i = s, f \qquad (a)\\ &x = 1: \theta_{i,tr} = 0; \quad (\partial^2\theta_{i,tr}/\partial x^2)_{x=1} = 0 \quad \forall i = s, f \qquad (b) \end{aligned} \quad (43)$$

$$t = 0: (\theta_{i,tr})_{t=0} = \theta_0 - x; \quad (\partial\theta_{i,tr}/\partial t)_{t=0} = 0 \quad \forall i = s, f \quad (44)$$

The solution to Eq. (42) subject to the boundary and initial conditions (43) and (44) is obtained via separation of variables in the form of two equations for each phase as $\theta_{i,tr} = \phi_{in}(t)u_n(x)$ where the function $u_n(x)$ is identical for both phases because it satisfies the same equations and the same boundary conditions. The equation for the common eigenfunction $u_n(x)$ is identical to the one obtained in the previous section and is subject to the same homogeneous boundary conditions $(u_n)_{x=0,1} = 0$ leading inevitably to the same eigenfunction solution $u_n(x) = \sin(n\pi x)$. The equations for the eigenfunctions in the time domain is

$$Fo_q\frac{d^2\phi_{in}}{dt^2} + (1 + Fo_T\kappa_n^2)\frac{d\phi_{in}}{dt} + \kappa_n^2(1 + \kappa_n^2 Bh)\phi_{in} = 0 \quad \forall i = s, f. \quad (45)$$

Eq. (45) is identical to a linear damped oscillator (mechanical mass-spring-damper $m - K - c$, or electrical L-R-C circuit). A more convenient form of (45) is obtained after dividing it by Fo_q to yield

$$\frac{d^2\phi_{in}}{dt^2} + \nu_n\frac{d\phi_{in}}{dt} + \omega_n^2\phi_{in} = 0 \quad \forall i = s, f \quad (46)$$

where the specific damping coefficient ν_n and natural frequency ω_n are the parameters defined in Eqs. (33) and (34), respectively.

The dimensionless group that emerged from the definition of ω_n^2 in (34) as a combination of the bi-harmonic number Bh and the heat flux Fourier number Fo_q is $Bf = Bh/Fo_q = \alpha_s\alpha_f/\alpha_e^2$, where $\alpha_s = k_s/\gamma_s$ and $\alpha_f = k_f/\gamma_f$. In addition, the dimensionless group that emerged from the definition of ν_n in (33) as a combination of the heat flux and temperature gradient related Fourier numbers Fo_q and Fo_T, respectively, is $\psi = Fo_T/Fo_q = \tau_T/\tau_q = 1 + Bf + (\eta_\gamma - \eta_k)^2/[\eta_\gamma\eta_k(1+\eta_k)(1+\eta_k^{-1})] \geqslant 1 + Bf > 1$, where $\eta_\gamma = \gamma_f/\gamma_s$ and $\eta_k = k_f/k_s$. Despite the similarity of Eq. (46) to a linear damped oscillator, physical constraints allow only overdamped solutions to exist in this particular application as demonstrated by Vadasz [13–15].

From (46), the equation for the eigenvalues has the form $\lambda_n^2 + \nu_n\lambda_n + \omega_n^2 = 0$, leading to the eigenvalues solutions

$$\lambda_{1n} = -\frac{\nu_n}{2}\left[1 + \sqrt{1 - 4\frac{\omega_n^2}{\nu_n^2}}\right] \quad \text{and} \quad \lambda_{2n} = -\frac{\nu_n}{2}\left[1 - \sqrt{1 - 4\frac{\omega_n^2}{\nu_n^2}}\right] \quad (47)$$

and the eigenfunctions ϕ_{in} are the superposition of $\exp[\lambda_{1n}t]$ and $\exp[\lambda_{2n}t]$

$$\phi_{in}(t) = A_{in}e^{\lambda_{1n}t} + B_{in}e^{\lambda_{2n}t} \quad \forall i = \mathrm{s,f} \tag{48}$$

leading to the solution for $\theta_{i,\mathrm{tr}}$ expressed in the form

$$\theta_{i,\mathrm{tr}} = \sum_{n=1}^{\infty} \phi_{in}(t)\sin(n\pi x) \quad \forall i = \mathrm{s,f} \tag{49}$$

The solution (49) includes two sequences of coefficients presented in (48) that need to be established from the two initial conditions (44) at $t = 0$. The first initial condition produces

$$(\theta_{i,\mathrm{tr}})_{t=0} \equiv \sum_{n=1}^{\infty} \phi_{in}(0)\sin(n\pi x) = \theta_0 - x \quad \forall i = \mathrm{s,f} \tag{50}$$

A relationship between A_{in} and B_{in} in Eq. (48) is obtained by multiplying Eq. (50) by $\sin(j\pi x)$, integrating the results over the whole domain, i.e. $\int_0^1(\cdot)\,\mathrm{d}x$ and using orthogonality conditions to yield

$$A_{in} + B_{in} = \phi_{no} \quad \forall i = \mathrm{s,f} \tag{51}$$

where

$$\phi_{sn}(0) = \phi_{\mathrm{f}n}(0) = \phi_{no} = \frac{2[\theta_0 + (1-\theta_0)(-1)^n]}{n\pi} \tag{52}$$

and from (52) the relationship between the coefficients A_{in} and B_{in} is

$$B_{in} = \phi_{no} - A_{in} \quad \forall i = \mathrm{s,f} \tag{53}$$

which upon substitution into (48) and then into (49) yields

$$\theta_{i,\mathrm{tr}} = \sum_{n=1}^{\infty}\left[A_{in}e^{\lambda_{1n}t} + (\phi_{no} - A_{in})e^{\lambda_{2n}t}\right]\sin(n\pi x) \quad \forall i = \mathrm{s,f} \tag{54}$$

Using now the second initial condition from (44) into (54) produces the equation

$$\left(\frac{\partial\theta_i}{\partial t}\right)_{t=0} \equiv \sum_{n=1}^{\infty}[A_{in}\lambda_{1n} + (\phi_{no} - A_{in})\lambda_{2n}]\sin(n\pi x) = 0 \quad \forall i = \mathrm{s,f} \tag{55}$$

The values of the coefficients A_{in} are finally obtained from (55) to yield

$$A_{in} = \frac{\lambda_{2n}\phi_{no}}{(\lambda_{2n} - \lambda_{1n})} \quad \forall i = \mathrm{s,f} \tag{56}$$

Eq. (56) indicates that the coefficients for both phases are identical, i.e. $A_{sn} = A_{\mathrm{f}n}$, a fact that causes the solutions for both phases to be identical too. The complete solution is obtained from (49) by substituting Eqs. (53) and (56) leading to

$$\theta_i = x + \sum_{n=1}^{\infty}\left[\frac{\lambda_{2n}\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}e^{\lambda_{1n}t} - \frac{\lambda_{1n}\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}e^{\lambda_{2n}t}\right]\sin(n\pi x) \quad \forall i = \mathrm{s,f} \tag{57}$$

and is perfectly consistent with the conclusion reached in Section 2, Eq. (9), indicating that the temperature of both phases are identical leading to Lotheq and consequently to the stated paradox.

5. Resolution of the paradox

While the eigenvalues obtained via both the elimination and the eigenvectors methods are identical leading to identical final forms of the solution let us compare the final coefficients in these solutions obtained via the two different methods. Comparing the coefficients of the $e^{\lambda_{1n}t}$ term in (37) and (38) with the corresponding coefficients to the same term in (57) shows that the first part of the coefficients is identical but the second part is missing in Eq. (57). Similarly for the coefficients to the $e^{\lambda_{2n}t}$ term, their second part is missing in Eq. (57).

What is therefore the reason that the elimination method produces an incorrect result? The first part of the answer to this question can be obtained by observing that both methods produce identical solutions up to the point where we imposed the second initial condition on the elimination method solution, Eq. (54). Only after imposing the initial condition (55) specifying a vanishing initial temperature derivative in time, i.e. $(\partial\theta_i/\partial t)_{t=0} = 0\ \forall i = \mathrm{s,f}$ the two solutions obtained via the two different methods diverged producing the apparent paradox. Then, the second part of the answer should be related to the question of why a perfectly correct initial condition obtained correctly from the analysis preceding Eqs. (7) and (8) produces an incorrect solution. The answer to this second part of the question is related to the way the coefficients were evaluated from the Fourier series in Eq. (55) by using this derivative initial condition. The implied assumption when doing so is that any constant (including the 0) can be expanded into a Fourier series. It is however naïve to expect the existence of a Fourier expansion to the 0 constant as Eq. (55) implies. The coefficients obtained this way are therefore incorrect, although the initial condition $(\partial\theta_i/\partial t)_{t=0} = 0\ \forall i = \mathrm{s,f}$ is indeed correct. In order to correct this evaluation of the coefficients via the elimination method let us check what do we need to do instead of using the initial condition $(\partial\theta_i/\partial t)_{t=0} = 0\ \forall i = \mathrm{s,f}$. We still need derivative initial conditions for $\phi_{in}(t)$, i.e. $(\mathrm{d}\phi_{in}/\mathrm{d}t)_{t=0}\ \forall i = \mathrm{s,f}$ in order to establish the value of the coefficients $A_{in}\ \forall i = \mathrm{s,f}$. However, as distinct from $(\partial\theta_i/\partial t)_{t=0}$ which needs the full system of partial differential equations (12), (13) to extract its value, the values of $(\mathrm{d}\phi_{in}/\mathrm{d}t)_{t=0}$ can be obtained from the system of ordinary differential equations (21a,b) by using the known values of $\phi_{sn}(0) = \phi_{\mathrm{f}n}(0) = \phi_{no}$ that were evaluated and presented in Eq. (52). Substituting these values for $t = 0$ in Eq. (21) yields

$$\left(\frac{\mathrm{d}\phi_{sn}}{\mathrm{d}t}\right)_{t=0} = a_n\phi_{sn}(0) + b\phi_{\mathrm{f}n}(0) = (a_n + b)\phi_{no} \qquad \text{(a)}$$
$$\left(\frac{\mathrm{d}\phi_{\mathrm{f}n}}{\mathrm{d}t}\right)_{t=0} = c\phi_{sn}(0) + d_n\phi_{\mathrm{f}n}(0) = (c + d_n)\phi_{no} \qquad \text{(b)}$$
$$\tag{58}$$

Now, from (54) we evaluated $\phi_{in}(t)$ up to the yet unknown value of the constants A_{in}, in the form

$$\phi_{in}(t) = A_{in}\mathrm{e}^{\lambda_{1n}t} + (\phi_{no} - A_{in})\mathrm{e}^{\lambda_{2n}t} \quad \forall i = \mathrm{s,f} \tag{59}$$

From (59) one can take the time derivative to yield

$$\frac{\mathrm{d}\phi_{in}}{\mathrm{d}t} = \lambda_{1n}A_{in}\mathrm{e}^{\lambda_{1n}t} + \lambda_{2n}(\phi_{no} - A_{in})\mathrm{e}^{\lambda_{2n}t} \quad \forall i = \mathrm{s,f} \tag{60}$$

and evaluating (60) at $t = 0$ produces

$$\left(\frac{\mathrm{d}\phi_{in}}{\mathrm{d}t}\right)_{t=0} = \lambda_{1n}A_{in} + \lambda_{2n}(\phi_{no} - A_{in}) \tag{61}$$

Substituting now the initial conditions (58) into (61) leads to the result

$$\lambda_{1n}A_{\mathrm{s}n} + \lambda_{2n}(\phi_{no} - A_{\mathrm{s}n}) = (a_n + b)\phi_{no} \tag{62}$$

$$\lambda_{1n}A_{\mathrm{f}n} + \lambda_{2n}(\phi_{no} - A_{\mathrm{f}n}) = (c + d_n)\phi_{no} \tag{63}$$

The values of the coefficients $A_{\mathrm{s}n}$ and $A_{\mathrm{f}n}$ can now be evaluated from (62) and (63) in the form

$$A_{\mathrm{s}n} = \frac{[\lambda_{2n} - (a_n + b)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})} \tag{64}$$

$$A_{\mathrm{f}n} = \frac{[\lambda_{2n} - (d_n + c)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})} \tag{65}$$

Substituting these results (64), (65) into (59) and the result into (49) and adding the steady part yields

$$\theta_\mathrm{s} = x + \sum_{n=1}^{\infty}\left[\frac{[\lambda_{2n} - (a_n + b)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{1n}t} - \frac{[\lambda_{1n} - (a_n + b)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{2n}t}\right]\sin(n\pi x) \tag{66}$$

$$\theta_\mathrm{f} = x + \sum_{n=1}^{\infty}\left[\frac{[\lambda_{2n} - (d_n + c)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{1n}t} - \frac{[\lambda_{1n} - (d_n + c)]\phi_{no}}{(\lambda_{2n} - \lambda_{1n})}\mathrm{e}^{\lambda_{2n}t}\right]\sin(n\pi x) \tag{67}$$

The first observation from the solutions (66) and (67) is that these solutions are not anymore identical, i.e. now we obtained LaLotheq conditions i.e. $\theta_\mathrm{s} \neq \theta_\mathrm{f}$ as we did via the eigenvector method. Comparing now these solutions obtained via the elimination method (66) and (67) with the solutions (37) and (38), respectively, obtained via the eigenvectors method by looking at the coefficients of the terms $\mathrm{e}^{\lambda_{1n}t}$ and $\mathrm{e}^{\lambda_{2n}t}$, brings us to the conclusion that both methods yield identical solutions, i.e. (37) is identical to (66) and (38) is identical to (67). The latter conclusion resolves therefore the paradox.

However, now that we have obtained identical solutions via both methods and resolved the paradox, it is interesting to observe how the initial temperature derivative with respect to time $(\partial\theta_i/\partial t)_{t=0}$ as evaluated from these solutions looks like, and whether it indeed vanishes as expected. Taking the time derivative of the solutions Eqs. (37), (38) or (66) and (67) and evaluating it at $t = 0$ leads to

$$\left(\frac{\partial\theta_\mathrm{s}}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty}\phi_{no}(a_n + b)\sin(n\pi x) \tag{68}$$

$$\left(\frac{\partial\theta_\mathrm{f}}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty}\phi_{no}(d_n + c)\sin(n\pi x) \tag{69}$$

Substituting the definitions of a_n, b, d_n and c from (22) and by using (14) yields

$$a_n + b = -\frac{n^2\pi^2}{Ni_\mathrm{s}Fh_\mathrm{s}} = -\frac{\alpha_\mathrm{s}}{\alpha_\mathrm{e}}n^2\pi^2 \tag{70}$$

$$d_n + c = -\frac{n^2\pi^2}{Ni_\mathrm{f}Fh_\mathrm{f}} = -\frac{\alpha_\mathrm{f}}{\alpha_\mathrm{e}}n^2\pi^2 \tag{71}$$

where $\alpha_\mathrm{s} = k_\mathrm{s}/\gamma_\mathrm{s}$, $\alpha_\mathrm{f} = k_\mathrm{f}/\gamma_\mathrm{f}$ and α_e is defined in Eq. (6). Substituting (70) and (71) as well as the value of ϕ_{no} from (52) into (68) and (69) yields

$$\left(\frac{\partial\theta_\mathrm{s}}{\partial t}\right)_{t=0} = -\frac{2\alpha_\mathrm{s}\pi}{\alpha_\mathrm{e}}\sum_{n=1}^{\infty}n[\theta_0 + (1 - \theta_0)(-1)^n]\sin(n\pi x) \tag{72}$$

$$\left(\frac{\partial\theta_\mathrm{f}}{\partial t}\right)_{t=0} = -\frac{2\alpha_\mathrm{f}\pi}{\alpha_\mathrm{e}}\sum_{n=1}^{\infty}n[\theta_0 + (1 - \theta_0)(-1)^n]\sin(n\pi x) \tag{73}$$

The simplest case is obtained for $\theta_0 = 0$ when (72) and (73) become

$$\left(\frac{\partial\theta_\mathrm{s}}{\partial t}\right)_{t=0} = -\frac{2\alpha_\mathrm{s}\pi}{\alpha_\mathrm{e}}\sum_{n=1}^{\infty}(-1)^n n\sin(n\pi x) \tag{74}$$

$$\left(\frac{\partial\theta_\mathrm{f}}{\partial t}\right)_{t=0} = -\frac{2\alpha_\mathrm{f}\pi}{\alpha_\mathrm{e}}\sum_{n=1}^{\infty}(-1)^n n\sin(n\pi x) \tag{75}$$

For any fixed value of x these alternating series have the form $\sum_{n=1}^{\infty}(-1)^n n = -1 + 2 - 3 + 4 - 5 + 6 - \cdots$, because the *sine* function will vary between -1 and 1 as the value of n changes at fixed x. The sum of any two consecutive terms is either 1 or -1, depending on the choice of grouping the terms. In both cases the sum becomes $\sum_{n=1}^{\infty}(-1)^n n = \sum_{n=1}^{\infty} \pm 1 \to \pm\infty$, hence we conclude that the series in (74) and (75) diverge and we cannot estimate $(\partial\theta_i/\partial t)_{t=0}$ analytically from the solutions. Actually the funny part of this result is that according to (74) and (75) the initial temperature derivatives $(\partial\theta_i/\partial t)_{t=0}$ are identically zero on the boundaries, at $x = 0, 1$, where we could have anticipated the singularity because of the temperature step change there. Yet the results show that $(\partial\theta_i/\partial t)_{t=0}$ is identically zero on the boundaries and diverges elsewhere. This particular anomaly should be the subject of further investigation.

6. Results and discussion

The analytical series solution obtained via both methods was evaluated and plotted in order to visualize the behavior of the solutions for both phases during the transient, evaluate the temperature differences between the phases and verify the analytical conclusions drawn.

The initial temperature value was taken as $\theta_0 = 0.5$, implying $T_0 = (T_H + T_C)/2$. The values of the parameters used were $Ni_f = 1$, $Ni_s = 0.5$ and $Fh_f = Fh_s = 1.5$.

The results are presented graphically in Fig. 2 in terms of θ_s and θ_f as a function of time at constant values of x. Fig. 2a presents the results for $x = 0.1$, 0.2, 0.3, and 0.4, while Fig. 2b presents the results for $x = 0.6$, 0.7, 0.8, and 0.9. It is obvious from these results that the temperatures of the phases are distinct, they start initially at $t = 0$ being identical, i.e. $(\theta_s)_{t=0} = (\theta_f)_{t=0} = \theta_0 = 0.5$ and they end-up being identical at steady state, i.e. $(\theta_s)_{t\to\infty} = (\theta_f)_{t\to\infty} = x$, but during the transient $\theta_s \neq \theta_f$.

These results are plotted in Fig. 3a in more detail while zooming into the initial time domain $t \in [0, 0.002]$ in order to check the initial time derivative of temperature $(\partial\theta_s/\partial t)_{t=0}$ and $(\partial\theta_f/\partial t)_{t=0}$. It is observed as anticipated that the temperature values overlap showing no variation in time except for the neighborhood of the boundaries, i.e. for $x = 0.1$ and $x = 0.9$. Even for this neighborhood it may be observed that there is an initial time domain $t \in [0, 0.0005]$ where temperature variations in time seem non-existent reinforcing the analytical conclusion that $(\partial\theta_s/\partial t)_{t=0} = (\partial\theta_f/\partial t)_{t=0} = 0$. The numerical values (not shown here) confirm this result to machine precision.

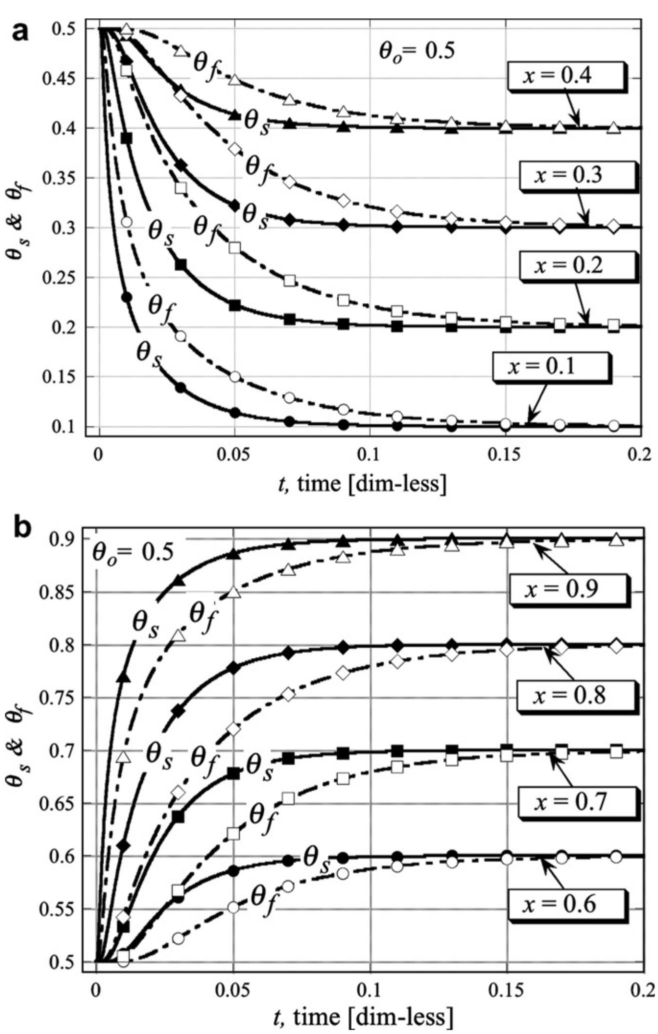

Fig. 2. Results of the analytical solution for the temperature of both phases as a function of time at selected locations: (a) at values of $x = 0.1, 0.2, 0.3$, and 0.4; (b) at values of $x = 0.6, 0.7, 0.8$ and 0.9.

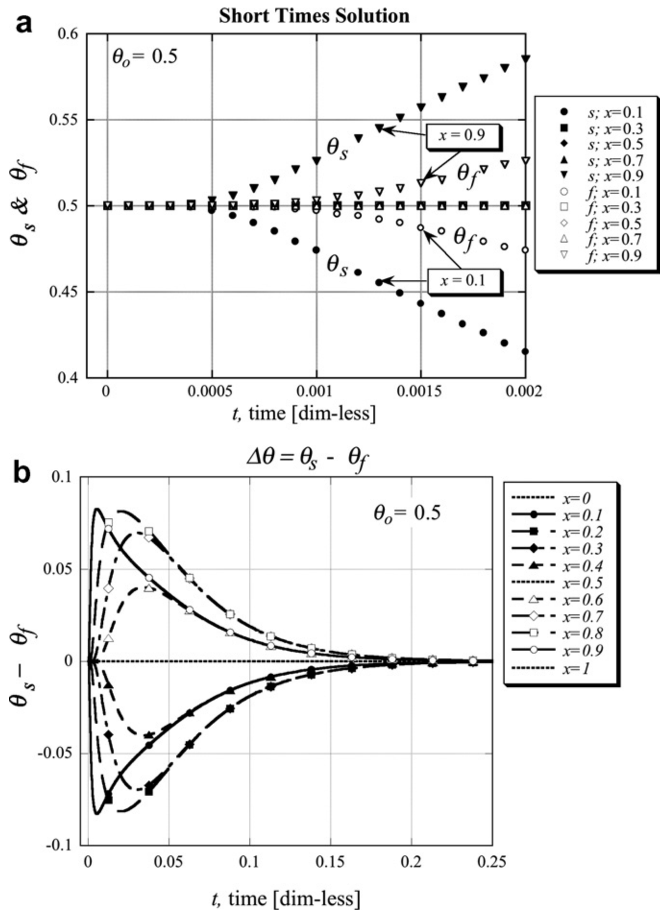

Fig. 3. Results of (a) the temperature solutions for short times by zooming into the initial time domain $t \in [0, 0.002]$ in order to check the initial time derivative of the temperature $(\partial\theta_s/\partial t)_{t=0}$ and $(\partial\theta_f/\partial t)_{t=0}$, (b) the temperature difference between the phases in terms of $\Delta\theta = (\theta_s - \theta_f)$ as a function of time at selected constant values of x.

The temperature difference between the phases in terms of $\Delta\theta = (\theta_s - \theta_f)$ as a function of time at selected constant values of x is presented in Fig. 3b clearly identifying the variation of the temperature difference between the phases with time, starting from and ending with identical values.

7. Conclusions

An apparent paradox that appears in problems of heat conduction in porous media subject to lack of local thermal equilibrium (LaLotheq) was reformulated and resolved. This apparent paradox relates to a combination of Dirichlet and insulation boundary conditions and leads the solution towards local thermal equilibrium (Lotheq). While the formulation, analysis and demonstration of the apparent paradox and its resolution was undertaken here for a specific two-dimensional and rectangular geometry, its generalization to an arbitrary three-dimensional geometry as presented by Vadasz [1] is not straightforward because the way around the Paradox in this particular case was approached via the Fourier decomposition, while the introduction of the general formulation of the Paradox in Vadasz [1] did not use such a decomposition and therefore the resolution applied in the particular case presented here cannot apply in the more general case formulation. We also identified as the source of the Paradox the fact that the initial temperature derivative with respect to time cannot be estimated from the analytical solution and actually produces an analytical anomaly. More work is needed to understand the reason for the latter and how this may affect the formulated Paradox.

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Appendix

Let us present Eq. (26) for the eigenvalues λ_{1n} and λ_{2n} in the form

$$\lambda_{1n} = \frac{(a_n + d_n)}{2} + \frac{\delta_n}{2}; \quad \lambda_{2n} = \frac{(a_n + d_n)}{2} - \frac{\delta_n}{2} \tag{A.1}$$

where $\delta_n = \sqrt{(a_n + d_n)^2 - 4(a_n d_n - bc)}$. Then by using (A.1)

$$(\lambda_{1n} - d_n) = \frac{(a_n + d_n)}{2} + \frac{\delta_n}{2} - d_n = \frac{(a_n - d_n)}{2} + \frac{\delta_n}{2} = \frac{1}{2}(a_n - d_n + \delta_n) \tag{A.2}$$

$$(\lambda_{2n} - a_n) = \frac{(a_n + d_n)}{2} - \frac{\delta_n}{2} - a_n = -\frac{(a_n - d_n)}{2} - \frac{\delta_n}{2} = -\frac{1}{2}(a_n - d_n + \delta_n) \tag{A.3}$$

Comparing (A.2) with (A.3) leads to the conclusion

$$(\lambda_{1n} - d_n) = -(\lambda_{2n} - a_n) \tag{A.4}$$

This result is the relationship presented in Eq. (30) in the text.

References

[1] P. Vadasz, Explicit conditions for Local Thermal Equilibrium in porous media heat conduction, Transp. Porous Media 59 (2005) 341–355.
[2] D.A. Nield, A note on the modeling of local thermal non-equilibrium in a structured porous medium, Int. J. Heat Mass Transfer 45 (2002) 4367–4368.
[3] D.A. Nield, Effects of local thermal nonequilibrium in steady convective processes in a saturated porous medium: forced convection in a channel, J. Porous Media 1 (1998) 181–186.
[4] W.J. Minkowycz, A. Haji-Sheikh, K. Vafai, On departure from Local Thermal Equilibrium in porous media due to a rapidly changing heat source: the Sparrow number, Int. J. Heat Mass Transfer 42 (1999) 3373–3385.
[5] N. Banu, D.A.S. Rees, Onset of Darcy–Benard convection using a thermal non-equilibrium model, Int. J. Heat Mass Transfer 45 (2002) 2221–2238.
[6] A.C. Baytas, I. Pop, Free convection in a square porous cavity using a thermal nonequilibrium model, Int. J. Thermal Sci. 41 (2002) 861–870.
[7] S.J. Kim, S.P. Jang, Effects of Darcy number, the Prandtl number, and the Reynolds number on Local Thermal Non-Equilibrium, Int. J. Heat Mass Transfer 45 (2002) 3885–3896.
[8] D.A.S. Rees, Vertical free convective boundary-layer flow in a porous medium using a thermal nonequilibrium model: elliptical effects, Z. Angew. Math. Phys. 53 (2002) 1–12.
[9] B. Alazmi, K. Vafai, Constant wall heat flux boundary conditions in porous media under Local Thermal Non-Equilibrium conditions, Int. J. Heat Mass Transfer 45 (2002) 3071–3087.
[10] D.A. Nield, A.V. Kuznetsov, M. Xiong, Effect of Local Thermal Non-Equilibrium on thermally developing forced convection in a porous medium, Int. J. Heat Mass Transfer 45 (2002) 4949–4955.
[11] D.Y. Tzou, Macro-to-Microscale Heat Transfer; The Lagging Behavior, Taylor & Francis, Washington, DC, 1997.
[12] D.Y. Tzou, A unified field approach for heat conduction from macro-to-micro-scales, ASME J. Heat Transfer 117 (1995) 8–16.
[13] P. Vadasz, Absence of oscillations and resonance in porous media Dual-Phase-Lagging Fourier heat conduction, J. Heat Transfer 127 (2004) 307–314.
[14] P. Vadasz, Lack of oscillations in Dual-Phase-Lagging heat conduction for a porous slab subject to imposed heat flux and temperature, Int. J. Heat Mass Transfer 48 (2005) 2822–2828.
[15] P. Vadasz, Exclusion of oscillations in heterogeneous and bi-composite media thermal conduction, Int. J. Heat Mass Transfer 49 (2006) 4886–4892.
[16] P. Vadasz, Heat conduction in nanofluid suspensions, J. Heat Transfer 128 (2006) 465–477.
[17] D.A. Nield, A. Bejan, Convection in Porous Media, third ed., Springer, New York, 2006.